

Universal sextic effective interaction at criticality

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The renormalization group approach in three dimensions is used to estimate the universal critical value of dimensionless sextic effective coupling constant for the Ising model. Four-loop expansion for g_6 is calculated and resummed resulting in $g_6^* = 1.596$, while the most accurate estimate for g_6^* is argued to be equal to 1.61.

The critical thermodynamics of the three-dimensional Ising model is known to be described by Euclidean scalar field theory with the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2} m_0^2 \varphi^2 + \frac{1}{2} (\nabla \varphi)^2 + \lambda \varphi^4 \right], \quad (1)$$

where a bare mass squared m_0^2 is proportional to $T - T_c^{(0)}$, $T_c^{(0)}$ being the phase transition temperature in the absence of the order parameter fluctuations. Taking fluctuations into account results in renormalizations of the mass $m_0 \rightarrow m$, the field $\varphi = \varphi_R \sqrt{Z}$, and the coupling constant $\lambda = m Z_4 Z^{-2} g_4$. Moreover, thermal fluctuations give rise to many-point correlations $\langle \varphi(x_1) \varphi(x_2) \dots \varphi(x_{2k}) \rangle$ and, correspondingly, to higher-order terms in the expansion of the free energy ("effective action") in powers of magnetization M :

$$F(M) - F(0) = \frac{1}{2} m^2 Z^{-1} M^2 + m Z^{-2} g_4 M^4 + \sum_{k=3}^{\infty} m^{3-k} Z^{-k} g_{2k} M^{2k}. \quad (2)$$

In the critical region, where fluctuations are so strong that they completely screen out the initial (bare) interaction, the behaviour of the system becomes universal and dimensionless effective couplings g_{2k} approach their asymptotic limits g_{2k}^* , i.e. they assume constant values which are also universal.

Recently, several attempts have been made to find numerical values of higher-order coupling constants g_6 , g_8 , etc. at criticality; particular attention has been paid to the sextic effective interaction g_6 . C. Bagnuls and C. Bervillier solving exact renormalization group equations in the local potential approximation have obtained $g_6^* = 2.40$ [1]. M. M. Tsylin has performed extensive Monte Carlo simulations of the Ising model behaviour in an external magnetic field and found $g_6^* = 2.05 \pm 0.15$ [2]. T. Reisz has deduced the estimate $g_6^* = 1.92 \pm 0.24$ analyzing linked cluster expansions for $O(n)$ -symmetric lattice model in the Ising case $n = 1$ [3].

These estimates obtained by means of essentially different methods are seen to be considerably scattered. Hence, the question about the numerical value of g_6^* remains, in fact,

open. On the other hand, there is a theoretical tool which has proven to be very efficient when used to study the critical behaviour of various phase transition models. We mean the renormalization group (RG) approach in three dimensions. This method has enabled one to calculate critical exponents and fixed point locations with an accuracy of order of 1 per cent for simple $O(n)$ -symmetric models [4,5,6] as well as for more complicated systems possessing two [7,8,9] and three [10,11] quartic coupling constants in their Landau–Wilson Hamiltonians. It is quite reasonable therefore to employ the field-theoretical RG approach in three dimensions for calculation of g_6^* and other universal higher-order couplings.

In this Letter, we estimate g_6^* calculating RG series for g_6 and applying Pade–Borel resummation technique to get proper numerical results. Actually, the first step in this direction has been already done by one of the authors [12] who has found two-loop RG expansion for g_6 which yielded $g_6^* = 1.50$. This estimate, however, is undoubtedly very crude. Indeed, it has been obtained on the base of the lower-order perturbative expansion within the theory having no small parameter while accurate enough numerical estimates, as is well known, may be extracted only from sufficiently long RG series. Below we find the RG expansion for g_6 in the four-loop approximation which will be shown to provide fair numerical estimate for the quantity of interest.

The method of calculating g_6 we use here is straightforward. Since in three dimensions higher-order bare couplings are irrelevant in RG sense (see, e.g., Ref. [13]), renormalized perturbative series for g_6 can be obtained from conventional Feynman graph expansion of this quantity in terms of the only bare coupling constant $-\lambda$. In its turn, λ may be expressed perturbatively as a function of renormalized dimensionless quartic coupling constant g_4 . Substituting corresponding power series for λ into original expansion we can obtain the RG series for g_6 . First two terms of this series have been presented earlier [12]. Thus, what we should find is the three-loop and four-loop contributions. As can be shown, they are formed

by 16 and 94 one-particle irreducible Feynman graphs, respectively. Their calculation gives:

$$g_6 = \frac{9}{\pi} \left(\frac{\lambda Z^2}{m} \right)^3 \left[1 - \frac{33}{2\pi} \frac{\lambda Z^2}{m} + 20.53966666 \left(\frac{\lambda Z^2}{m} \right)^2 - 73.41441479 \left(\frac{\lambda Z^2}{m} \right)^3 \right], \quad (3)$$

The perturbative expansion for λ emerges directly from the normalizing condition $\lambda = mZ_4Z^{-2}g_4$ and the well-known expansion for Z_4 :

$$Z_4 = 1 + \frac{9}{2\pi} g_4 + \frac{63}{4\pi^2} g_4^2 + 1.778667825 g_4^3 + O(g_4^4). \quad (4)$$

Combining these expressions we obtain

$$g_6 = \frac{9}{\pi} g_4^3 \left(1 - \frac{3}{\pi} g_4 + 1.389962951 g_4^2 - 2.50173240 g_4^3 \right). \quad (5)$$

Being a field-theoretical perturbative expansion, this series is divergent (asymptotic). Hence, direct substitution of the fixed point value g_4^* into (5) would not lead to satisfactory results. To get reasonable numerical estimate for g_6^* some procedure making this expansion convergent should be applied. Since the series (5) is alternating simple Pade-Borel technique may play a role of such a procedure. Pade approximants of $[L/1]$ type, when used for analytical continuation of Borel transforms, are known to provide rather good results for various Landau-Wilson models (see, e.g., Refs. [4,6,10,11]). With the expansion (5) in hand, we can construct Pade approximant $[2/1]$ for its Borel transform $F(y)$ which is related to the function to be found ("sum of series") by the formula

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \int_0^{\infty} e^{-t} F(xt) dt, \quad F(y) = \sum_{k=0}^{\infty} \frac{c_k}{k!} y^k. \quad (6)$$

Numerical estimate for g_6^* is then obtained by evaluation of the Borel integral under $g_4 = g_4^*$ where high-precision fixed point value of g_4 is known from six-loop RG calculations: $g_4^* = 0.988$ [4,5]. The final result is as follows:

$$g_6^* = 1.596. \quad (7)$$

How close to the exact value of g_6^* may this number be? To clear up this point let's compare the estimates given by four subsequent RG approximations. One-, two-, three-, and four-loop calculations of g_6^* give 2.763, 1.500, 1.622, and 1.596, respectively (the three-loop estimate has been presented earlier in Ref. [14]). Since this set of numbers demonstrates oscillatory (and rapid!) convergence one may expect that exact sextic effective coupling constant lies between the three-loop and four-loop RG estimates. So, our four-loop RG analysis leading to the number (7) underestimates g_6^* by less than 2%. Moreover, it is possible to further improve the estimate for g_6^* by addressing the average

$$\frac{1.622 + 1.596}{2} = 1.609 \approx 1.61, \quad (8)$$

which can be referred to as a most accurate approximate value differing from the exact one by no more than 1%.

Another source of information about the accuracy of numerical results is their sensitivity to the type of resummation procedure. To get such an information we calculate g_6^* using the Borel-Leroy transformation

$$f(x) = \int_0^\infty t^b e^{-t} F(xt) dt, \quad (9)$$

which contains a free parameter b . This parameter is chosen in a way that provides fastest possible convergence of the iteration scheme. Being applied to the RG expansion (5) the machinery described gives $g_6^* = 1.61$ under the optimal value of b . This number is seen to coincide with the estimate (8) what may be considered as a strong argument in favour of its high accuracy.

The value of g_6^* we have just determined turns out to differ substantially from those obtained in Refs. [1,2,3]. At the same time, it agrees fairly well with the estimate which follows from the three-loop ϵ -expansion for the ratio g_6/g_4^2 [15]:

$$\frac{g_6}{g_4^2} = 2\epsilon - \frac{20}{27}\epsilon^2 + 1.2759\epsilon^3 + O(\epsilon^4). \quad (10)$$

Indeed, resumming this expansion by the Pade–Borel method and then putting $\epsilon = 1$ and $g_4 = 0.988$ we find $g_6^* = 1.653$, i.e. the number which is sufficiently close to (8). Good agreement takes place also between our results and very fresh estimates $g_6^* = 1.63 \pm 0.05$ and $g_6^* = 1.57 \pm 0.10$ deduced from strong coupling series and high temperature expansions for the lattice model [16]. At last, numbers (7), (8) are in a perfect accord with another very recent estimate $g_6^* = 1.604$ [17] found by the analysis of the five-loop scaling equation of state [18] which has used more sophisticated than ours resummation technique (the Borel–Leroy transformation combined with a conformal mapping).

To summarize, we have calculated the four-loop RG expansion for dimensionless sextic effective coupling constant g_6 of the three-dimensional Ising model. Resummation of this expansion by the Pade–Borel method has lead at criticality to the result $g_6^* = 1.596$. Having analyzed the set of four subsequent RG estimates for g_6^* the number 1.61 has been argued to be the best approximate value of universal sextic effective coupling accurate within 1%. Precisely the same estimate has been found by applying another, Pade–Borel–Leroy procedure to resum the four-loop RG series. Such a stability of the obtained value under the change of resummation technique may be considered as a serious argument in favour of its high accuracy. Although $g_6^* = 1.61$ is substantially smaller than the early estimates [1,2,3] it turns out to be in a good agreement with that following from the Pade–Borel resummed ϵ -expansion and with the results of recent advanced calculations [16,17].

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